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On the classical $W_4^{(2)}$ algebra

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Abstract. We consider the classical $W_4^{(2)}$ algebra from the integrable system viewpoint. The integrable evolution equations associated with the $W_4^{(2)}$ algebra are constructed and the Miura maps and consequent modifications are presented. Modifying the Miura maps, we supply a free field realization for the classical $W_4^{(2)}$ algebra. We also construct the corresponding Toda-type integrable systems.

1. Introduction

Integrable systems of nonlinear differential equations have been studied extensively during the last three decades [10]. These are equations which possess remarkable analytical, geometric and algebraic properties. An even more remarkable fact is that this theory brings a number of diverse research fields together and finds applications in several branches. The interaction between integrable system theory and W algebra theory is just one of many fascinating points which have attracted much attention recently [4, 21, 12].

W_n algebra, which is a higher-spin generalization of Virasoro algebra, has been introduced recently by Zamolodchikov and colleagues [21]. It plays an important role in the theory of two-dimensional quantum gravity and matrix models. W_n algebra may be constructed via the Hamiltonian reduction approach from the WZW model [4]. Novel W algebras exist which involve fields with fractional spins. These are referred to as $W_n^{(l)}$ algebras. The first such example is the $W_3^{(2)}$ algebra of Polyakov–Bershadsky [19], which consists of four fields: the energy–momentum tensor, two bosonic fields of spin $\frac{3}{2}$ and a spin-1 $U(1)$ current.

Gervais [12] was the first to notice the inter-relation between the KdV equation and the Virasoro algebra. More precisely, the second Poisson bracket of the KdV equation is equivalent to the classical Virasoro algebra. This result has been generalized and the equivalence of the classical W_n algebra and the second Poisson bracket of the Gelfand–Dickey hierarchy discovered [15, 2]. Thus, a unified representation of the classical W_n algebra is available. This remarkable connection provides new insight into both theories. For example, constructing a new type of integrable system may lead to a new type of W algebra, and *vice versa*. Another important point is that the Miura map, which plays a central role in soliton theory, often provides free field realization for the corresponding W algebra. Noticing the connection of the W algebras and integrable evolution equations, it is

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not surprising that a correspondence between the W algebras and Toda-type systems exists. We refer to [4] for more details.

While we have a unified description of the classical W_n algebra by means of the Gelfand–Dickey bracket, such a result is still lacking for the $W_n^{(l)}$ algebra. Therefore, this algebra deserves to be studied for small n and l, aiming to get a better understanding of the general case. The present paper investigates the $W_4^{(2)}$ algebra. Generally speaking, $W_n^{(l)}$ algebras are given by different $sl(2)$ embedding into $sl(n)$. The form of the $W_4^{(2)}$ algebra is inferred in [1], but its explicit form is presented by Bakas and Depireux [3] by means of a Hamiltonian reduction method. We notice that Depireux and Mathieu [9] discussed the $W_n^{(l)}$ algebra from the integrable system viewpoint. Their method is to exchange the evolution parameter of the integrable equations. They shown that this method is successful for the classical $W_4^{(3)}$ algebra, but fails in the case of the classical $W_4^{(2)}$ algebra. We will fill this gap in this paper and our method starts with the specification of the spectral problem. Both nonlinear evolution equations and the Toda type of systems will be given explicitly for the $W_4^{(2)}$ algebra.

As is known, it is important to have a free field realization for a given W algebra so that one may quantize it. In many cases, the Miura map serves as a free field realization. We will see that the standard Miura map does not give us free field realization of the classical $W_4^{(2)}$ algebra, but its proper modification does. A by-product is the free field realization for the W algebra associated with a matrix Schrödinger operator.

The paper is arranged as follows. We recall the explicit form of the classical $W_4^{(2)}$ algebra in section 2. In section 3, we construct the hierarchy of nonlinear evolution equations for this algebra. Section 4 is intended to construct its free field realization. The integrable systems of Toda type are presented in section 5. The final section contains some comments.

For simplicity, we only give non-vanishing commutator relations for Poisson bracket algebras. Also, ∂ always denotes the partial derivative with respect to x and all the fields are functions of x except those specified explicitly.

2. The classical $W_4^{(2)}$ algebra

We recall the classical $W_4^{(2)}$ algebra in this section. It was presented by Bakas and Depireux [3] by means of a Hamiltonian reduction approach. This algebra involves seven fields with a spin content of $(\frac{1}{2}, 1, \frac{3}{2}, \frac{3}{2}, 2, 2, \frac{5}{2})$. In this paper, we are concerned with its twisted version:

$$
\begin{aligned}
&\{T(x), T(y)\} = (\partial^3 + T\partial + \partial T)\delta(x-y)\\
&\{T(x), v(y)\} = (\tfrac{1}{2}\partial^3 - \tfrac{3}{2}H\partial^2 + v\partial + \partial v + w\partial p - p\partial w - H_{xx} - 2H_x\partial)\delta(x-y)\\
&\{T(x), q(y)\} = (\tfrac{1}{2}\partial^2 w + q\partial + \partial q)\delta(x-y) \qquad \{T(x), p(y)\} = p\partial\delta(x-y)\\
&\{T(x), r(y)\} = (\tfrac{1}{2}\partial^2 p + r\partial + \partial r)\delta(x-y) \qquad \{T(x), w(y)\} = w\partial\delta(x-y)\\
&\{T(x), H(y)\} = H\partial\delta(x-y)\\
&\{v(x), v(y)\} = (\tfrac{3}{4}\partial^3 - \tfrac{3}{4}H\partial H + v\partial + \partial v - \tfrac{3}{4}H_{xx} - \tfrac{3}{2}H_x\partial)\delta(x-y)\\
&\{v(x), q(y)\} = (\tfrac{3}{4}\partial^2 w + \tfrac{3}{4}H\partial w + \partial q + Hq + wv)\delta(x-y)\\
&\{v(x), r(y)\} = (-p\partial^2 - \tfrac{1}{4}\partial^2 p + p\partial H - pv - \tfrac{1}{4}H\partial p + r\partial - rH)\delta(x-y)\\
&\{v(x), p(y)\} = (-p\partial + r)\delta(x-y) \qquad \{v(x), w(y)\} = -q\delta(x-y)\\
&\{v(x), H(y)\} = -\tfrac{1}{2}(\partial + H)\partial\delta(x-y) \qquad \{q(x), q(y)\} = -\tfrac{3}{4}w\partial w\delta(x-y)\\
&\{q(x), r(y)\} = (\partial^3 - \partial^2 H - H\partial^2 + \partial v + u\partial - H(u+v) + H\partial H + \tfrac{1}{4}w\partial p)\delta(x-y)
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
&\{q(x), p(y)\} = ((\partial - H)\partial - v + ut)\delta(x-y) \qquad \{q(x), w(y)\} = 0\\
&\{q(x), H(y)\} = (\tfrac{1}{2}w\partial - q)\delta(x-y) \qquad \{r(x), r(y)\} = -\tfrac{3}{4}p\partial p\delta(x-y)\\
&\{r(x), w(y)\} = ((\partial - H)\partial + v - u)\delta(x-y) \qquad \{r(x), p(y)\} = 0\\
&\{r(x), H(y)\} = \tfrac{1}{2}(-p\partial + r)\delta(x-y) \qquad \{w(x), w(y)\} = 0\\
&\{w(x), p(y)\} = -2(\partial - H)\delta(x-y) \qquad \{w(x), H(y)\} = -w\delta(x-y)\\
&\{p(x), p(y)\} = 0 \qquad \{p(x), H(y)\} = p\delta(x-y)\\
&\{H(x), H(y)\} = -\partial\delta(x-y).
\end{aligned}
$$

It is easy to see that the above algebra is conformal with the Virasoro generator T. The spin content of this version is now (1, 1, 1, 2, 2, 2, 2).

3. The integrable hierarchy of evolution equations

In order to derive an integrable hierarchy of nonlinear evolution equations associated with the $W_4^{(2)}$ algebra, we specify the associated spectral problem first. Our spectral problem is

$$
\Phi_x = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ u+\lambda & q & H+h & w \\ r & v+\lambda & p & -H+h \end{bmatrix} \Phi. \tag{3.1}
$$

The motivation for choosing the above form of the spectral problem comes from Bakas and Depireux [3]. We note that this spectral problem may be rewritten equivalently in Lax operator form:

$$
\boldsymbol{L}\psi \equiv \left(\partial^2 - \begin{bmatrix} H+h & w \\ p & -H+h \end{bmatrix} + \begin{bmatrix} u & q \\ r & v \end{bmatrix}\right)\psi = -\lambda\psi. \tag{3.2}
$$

This is nothing but the matrix Schrödinger problem. With (3.1) or (3.2) we may use the standard approaches to construct the associated flows and Hamiltonian (Poisson) structures. Indeed, we may either follow the method described in [7] to calculate the two Poisson tensors or construct the matrix generalized Gelfand–Dickey brackets as in [11, 5]. Since this calculation is straightforward, we just list the results

$$
f_{t_n} = \{f, \mathcal{H}_{n+1}\}_0 = \{f, \mathcal{H}_n\}_1 \qquad f = u, v, q, r, p, w, H, h.
$$

Here two Poisson brackets are defined by

$$
\begin{aligned}
&\{u(x), u(y)\}_0 = -2\partial\delta(x-y) \qquad \{u(x), q(y)\}_0 = w\delta(x-y)\\
&\{u(x), r(y)\}_0 = -p\delta(x-y) \qquad \{v(x), v(y)\}_0 = -2\partial\delta(x-y)\\
&\{v(x), q(y)\}_0 = -w\delta(x-y) \qquad \{v(x), r(y)\}_0 = p\delta(x-y)\\
&\{q(x), r(y)\}_0 = -2(\partial + H)\delta(x-y).
\end{aligned}
$$

All other brackets vanish and

$$
\begin{aligned}
&\{u(x), u(y)\}_1 = (\partial^3 - (H+h)\partial(H+h) + u\partial + \partial u + (H+h)_{xx} + 2(H+h)_x\partial)\delta(x-y)\\
&\{u(x), v(y)\}_1 = (-w\partial p - wr + qp)\delta(x-y)\\
&\{u(x), q(y)\}_1 = (-w\partial^2 - w\partial(H+h) - wu + q\partial + q(H+h))\delta(x-y)\\
&\{u(x), r(y)\}_1 = ((\partial - H - h)(\partial p + r) + up)\delta(x-y)\\
&\{u(x), w(y)\}_1 = (-w\partial + q)\delta(x-y) \qquad \{u(x), H(y)\}_1 = \tfrac{1}{2}(\partial - H - h)\partial\delta(x-y)\\
&\{u(x), p(y)\}_1 = -r\delta(x-y) \qquad \{u(x), h(y)\}_1 = \tfrac{1}{2}(\partial - H - h)\partial\delta(x-y)
\end{aligned}
$$

$$
\begin{aligned}
&\{v(x), v(y)\}_1 = (\partial^3 - (H-h)\partial(H-h) + v\partial + \partial v - (H-h)_{xx} - 2(H-h)_x\partial)\delta(x-y)\\
&\{v(x), q(y)\}_1 = ((\partial + H - h)(\partial w + q) + wv)\delta(x-y)\\
&\{v(x), r(y)\}_1 = (-p\partial^2 + p\partial(H-h) - pv + r(\partial - H + h))\delta(x-y)\\
&\{v(x), w(y)\}_1 = -q\delta(x-y) \qquad \{v(x), H(y)\}_1 = -\tfrac{1}{2}(\partial + H - h)\partial\delta(x-y)\\
&\{v(x), p(y)\}_1 = (-p\partial + r)\delta(x-y) \qquad \{v(x), h(y)\}_1 = \tfrac{1}{2}(\partial + H - h)\partial\delta(x-y) \qquad (3.3)\\
&\{q(x), q(y)\}_1 = -w\partial w\delta(x-y) \qquad \{q(x), w(y)\}_1 = 0\\
&\{q(x), r(y)\}_1 = [\partial^3 - \partial^2 H - H\partial^2 + h_{xx} + 2h_x\partial + \partial v + u\partial - H(u+v) - h(v-u)\\
&\qquad\qquad +(H+h)\partial(H-h)]\delta(x-y)\\
&\{q(x), p(y)\}_1 = ((\partial - H - h)\partial - v + u)\delta(x-y) \qquad \{q(x), h(y)\}_1 = -\tfrac{1}{2}w\partial\delta(x-y)\\
&\{q(x), H(y)\}_1 = (\tfrac{1}{2}w\partial - q)\delta(x-y) \qquad \{r(x), r(y)\}_1 = -p\partial p\delta(x-y)\\
&\{r(x), w(y)\}_1 = ((\partial + H - h)\partial + v - u)\delta(x-y) \qquad \{r(x), p(y)\}_1 = 0\\
&\{r(x), H(y)\}_1 = (-\tfrac{1}{2}p\partial + r)\delta(x-y) \qquad \{r(x), h(y)\}_1 = -\tfrac{1}{2}p\partial\delta(x-y)\\
&\{w(x), w(y)\}_1 = 0 \qquad \{w(x), p(y)\}_1 = 2(-\partial + H)\delta(x-y)\\
&\{w(x), H(y)\}_1 = -w\delta(x-y) \qquad \{w(x), h(y)\}_1 = 0 \qquad \{p(x), p(y)\}_1 = 0\\
&\{p(x), H(y)\}_1 = p\delta(x-y) \qquad \{p(x), h(y)\}_1 = 0\\
&\{H(x), H(y)\}_1 = -\partial\delta(x-y) \qquad \{H(x), h(y)\}_1 = 0 \qquad \{h(x), h(y)\}_1 = -\partial\delta(x-y).
\end{aligned}
$$

Hamiltonians may be calculated most easily from

$$\mathcal{H}_n = \frac{2}{n}\int \operatorname{tr}\operatorname{res}(\boldsymbol{L}^{n/2})\,\mathrm{d}x \qquad \forall n \geqslant 1$$

where tr and res mean taking matrix trace and the coefficient of the term ∂^{-1}, respectively. The operator is defined in equation (3.2).

The classical $W_4^{(2)}$ algebra comes into play with the following observation: if we do the reduction $h = 0$ for the Poisson algebra (3.4), we obtain

$$
\begin{aligned}
&\{u(x), u(y)\} = (\tfrac{3}{4}\partial^3 - \tfrac{3}{4}H\partial H + u\partial + \partial u + \tfrac{3}{4}H_{xx} + \tfrac{3}{2}H_x\partial)\delta(x-y)\\
&\{u(x), v(y)\} = (-w\partial p - wr + qp - \tfrac{1}{4}(\partial^3 - \partial^2 H - H\partial^2 + H\partial H))\delta(x-y)\\
&\{u(x), q(y)\} = (-w\partial^2 - w\partial H - wu + q\partial + qH - \tfrac{1}{4}(\partial - H)\partial w))\delta(x-y)\\
&\{u(x), r(y)\} = ((\partial - H)(\partial p + r) + up - \tfrac{1}{4}(\partial - H)\partial p)\delta(x-y)\\
&\{u(x), w(y)\} = (-w\partial + q)\delta(x-y) \qquad \{u(x), p(y)\} = -r\delta(x-y)\\
&\{u(x), H(y)\} = \tfrac{1}{2}(\partial - H)\partial\delta(x-y)\\
&\{v(x), v(y)\} = (\tfrac{3}{4}\partial^3 - \tfrac{3}{4}H\partial H + v\partial + \partial v - \tfrac{3}{4}H_{xx} - \tfrac{3}{2}H_x\partial)\delta(x-y)\\
&\{v(x), q(y)\} = (\tfrac{3}{4}\partial^2 w + \tfrac{3}{4}H\partial w + \partial q + Hq + wv)\delta(x-y)\\
&\{v(x), r(y)\} = -(p\partial^2 + \tfrac{1}{4}\partial^2 p - p\partial H + pv + \tfrac{1}{4}H\partial p - r\partial + rH)\delta(x-y)\\
&\{v(x), p(y)\} = (-p\partial + r)\delta(x-y) \qquad \{v(x), w(y)\} = -q\delta(x-y) \qquad (3.4)\\
&\{v(x), H(y)\} = -\tfrac{1}{2}(\partial + H)\partial\delta(x-y) \qquad \{q(x), q(y)\} = -\tfrac{3}{4}w\partial w\delta(x-y)\\
&\{q(x), r(y)\} = [\partial^3 - \partial^2 H - H\partial^2 + \partial v + u\partial - H(u+v) + H\partial H + \tfrac{1}{4}w\partial p]\delta(x-y)\\
&\{q(x), p(y)\} = ((\partial - H)\partial - v + u)\delta(x-y) \qquad \{q(x), w(y)\} = 0\\
&\{q(x), H(y)\} = (\tfrac{1}{2}w\partial - q)\delta(x-y)\\
&\{r(x), w(y)\} = ((\partial - H)\partial + v - u)\delta(x-y) \qquad \{r(x), r(y)\} = -\tfrac{3}{4}p\partial p\delta(x-y)\\
&\{r(x), H(y)\} = -\tfrac{1}{2}(p\partial - r)\delta(x-y) \qquad \{r(x), p(y)\} = 0
\end{aligned}
$$

$$\{w(x), p(y)\} = 2(-\partial + H)\delta(x-y) \qquad \{w(x), w(y)\} = 0$$
$$\{w(x), H(y)\} = -w\delta(x-y) \qquad \{p(x), p(y)\} = 0$$
$$\{p(x), H(y)\} = p\delta(x-y) \qquad \{H(x), H(y)\} = -\partial\delta(x-y).$$

This algebra is nothing but the classical $W_4^{(2)}$ algebra (2.1) with the fields redefined by

$$T = u + v - H^2 - pw \qquad v = v \qquad p = p \qquad w = w$$
$$H = H \qquad q = q \qquad r = r.$$

Remark. We should emphasize that the reduction involved here is the standard Dirac reduction (see, for example, [17]).

Thus, we rediscover the classical $W_4^{(2)}$ algebra from the viewpoint of integrable systems. Because of this equivalence, we also call the Poisson algebra (3.4) $W_4^{(2)}$. The explicit form of integrable hierarchy associated with it can be read off from the hierarchy (3). Here, we just give the first non-trivial flow:

$$\begin{aligned}
u_t &= \tfrac{1}{2}(-HH_x + H_{xx} + 2u_x - wp_x - wr + qp)\\
v_t &= \tfrac{1}{2}(-HH_x + H_{xx} + 2v_x - w_x p + wr + qp)\\
q_t &= \tfrac{1}{2}(-Hw_x + w_{xx} + 2q_x + wH_x - 2Hq + wu - wv)\\
r_t &= \tfrac{1}{2}(Hp_x + p_{xx} + 2r_x - pH_x - pu + pv + 2rH)\\
w_t &= p_t = H_t = 0.
\end{aligned} \tag{3.5}$$

Remark. We note that in the above system the time evolution of the fields (w, p, H) is trivial. This means that the dynamical system may be reduced to the submanifold of (u, v, q, r). In fact, this is a general phenomenon: the whole hierarchy is reducible to the submanifold (u, v, q, r) (see [5]).

4. The free field realization of the $W_4^{(2)}$ algebra

For a given W algebra, it is important to construct a free field realization. As is well known, the Miura-type map serves as a free field realization in many cases. Thus, to construct such a realization for our $W_4^{(2)}$ algebra (3.4), we start with the derivation of Miura maps for the related hierarchy.

Let us make the following factorization [11, 5]:

$$\boldsymbol{L} = (\partial - M)(\partial - N)$$

where $\boldsymbol{L}$ is the matrix Schrödinger operator given by (3.2), $M = \begin{bmatrix} g_1 & k \\ l & g_2 \end{bmatrix}$, $N = \begin{bmatrix} m_1 & n \\ s & m_2 \end{bmatrix}$. Then, the transformation between field variables, which is a Miura map, reads

$$\begin{aligned}
&u = g_1 m_1 + ks - m_{1x} \qquad v = ln + g_2 m_2 - m_{2x}\\
&q = g_1 n + k m_2 - n_x \qquad r = l m_1 + g_2 s - s_x \qquad w = k + n\\
&p = l + s \qquad H = \tfrac{1}{2}(g_1 + m_1 - g_2 - m_2) \qquad h = \tfrac{1}{2}(g_1 + g_2 + m_1 + m_2).
\end{aligned} \tag{4.1}$$

The modified Poisson bracket may be calculated from the bracket directly following [11, 5]. The resulting non-vanishing brackets are given by

$$\begin{aligned}
&\{m_i(x), m_i(y)\} = -\partial\delta(x-y) \qquad \{m_i(x), n(y)\} = \pm n\delta(x-y)\\
&\{m_i(x), s(y)\} = \mp s\delta(x-y) \qquad \{n(x), s(y)\} = (-\partial + m_1 + m_2)\delta(x-y)
\end{aligned} \tag{4.2a}$$

$$\begin{aligned}
&\{g_i(x), g_i(y)\} = -\partial\delta(x-y) \qquad \{g_i(x), k(y)\} = \pm k\delta(x-y)\\
&\{g_i(x), l(y)\} = \mp s\delta(x-y) \qquad \{k(x), l(y)\} = (-\partial + g_1 + g_2)\delta(x-y)
\end{aligned} \tag{4.2b}$$

where $i = 1, 2$.

It can be directly verified that the Miura map (4.1) is a Hamiltonian or Poisson map. That is, it maps the modified Poisson bracket ($4.2a, b$) to the Poisson bracket (3.3).

Up to now, all these are known and standard (see [11, 5]). However, we note that unlike the scalar case, the present Miura map (4.1) does not supply us with a free field realization for the Poisson algebra (3.3) although it does simplify this algebra greatly. To obtain such a realization, we need to introduce further coordinate transformations. Since our Lax operator has been factorized already, we are not able to obtain further transformations via the factorization. To proceed, we notice the special structure of the Poisson algebra ($4.2a, b$): it consists of two closed subalgebras ($4.2a$) and ($4.2b$). Thus, we only need to work on one of them, say the subspace (m_1, m_2, n, s) with the algebra ($4.2a$). Now, our observation is that the following transformation

$$\bar{m}_1 = m_1 + m_2 \qquad \bar{m}_2 = m_1 - m_2 \qquad \bar{n} = n \qquad \bar{s} = s \tag{4.3a}$$

maps ($4.2a$) to

$$\{\bar{m}_1(x), \bar{m}_1(y)\} = -2\partial\delta(x-y) \tag{4.4a}$$

$$\begin{aligned} &\{\bar{m}_2(x), \bar{m}_2(y)\} = -2\partial\delta(x-y) \qquad \{\bar{m}_2(x), \bar{n}(y)\} = 2\bar{n}\delta(x-y) \\ &\{\bar{m}_2(x), \bar{s}(y)\} = -2\bar{s}\delta(x-y) \qquad \{\bar{n}(x), \bar{s}(y)\} = (-\partial + \bar{m}_2)\delta(x-y). \end{aligned} \tag{4.4b}$$

The nice feature here is that the Poisson bracket algebra ($4.2a$) is decoupled by the transformation ($4.3a$). That is, the algebra ($4.4a, b$) consists of two parts: a $U(1)$ current m_1 and the $sl(2)$ current algebra ($4.4b$). Thus, we may use Wakimoto construction [20] to get the free field realization. The same device works with the other subspace (g_i, k, l). Summarizing, we have

$$\bar{m}_1 = \xi \qquad \bar{m}_2 = \sqrt{2}\alpha + 2\beta\gamma \qquad \bar{n} = -\beta\gamma^2 + \gamma_x - \sqrt{2}\gamma\alpha \qquad \bar{s} = \beta \tag{4.3b}$$

and

$$\bar{g}_1 = g_1 + g_2 \qquad \bar{g}_2 = g_1 - g_2 \qquad \bar{k} = k \qquad \bar{l} = l \tag{4.5a}$$

$$\bar{g}_1 = \zeta \qquad \bar{g}_2 = \sqrt{2}\mu + 2\eta\nu \qquad \bar{k} = -\nu\eta^2 + \eta_x - \sqrt{2}\mu\eta \qquad \bar{s} = \nu. \tag{4.5b}$$

The final bracket is defined in the coordinates $(\xi, \alpha, \gamma, \beta, \zeta, \mu, \eta, \nu)$ by

$$\begin{aligned} &\{\xi(x), \xi(y)\} = \{\zeta(x), \zeta(y)\} = -2\partial\delta(x-y) \\ &\{\alpha(x), \alpha(y)\} = \{\mu(x), \mu(y)\} = -\partial\delta(x-y) \\ &\{\gamma(x), \beta(y)\} = \{\eta(x), \nu(y)\} = -\delta(x-y). \end{aligned} \tag{4.6}$$

Thus, we reach the free field realization for the Poisson bracket algebra (3.3). We stress that the transformations (4.1), (4.3) and (4.5) together serve as the mapping to a free field realization.

Next we turn to our main object: the classical $W_4^{(2)}$ algebra. Since this Poisson algebra is obtainable from the more general one (3.3) by reduction, we expect that some kind of reduction or constraint of the Miura map (4.1) simplifies the algebra (3.4). Thus, let us first do the following recoordinating:

$$(m_1, m_2, n, s, g_1, g_2, k, l) \to (m_1, m_2, n, s, g_1, k, l, E) \tag{4.7}$$

where $E = m_1 + m_2 + g_1 + g_2$. Then we perform the Dirac reduction $E = 0$ for the transformed algebra. Under this new coordinate, the algebra becomes

$$\begin{aligned}
&\{m_i(x), m_i(y)\} = \{g_1(x), g_1(y)\} = -\tfrac{3}{4}\partial\delta(x-y)\\
&\{m_1(x), m_2(y)\} = \{m_i(x), g_1(y)\} = -\tfrac{1}{4}\partial\delta(x-y)\\
&\{m_i(x), n(y\} = \pm n\delta(x-y) \qquad \{m_i(x), s(y)\} = \mp s\delta(x-y)\\
&\{n(x), s(y)\} = (-\partial + m_1 + m_2)\delta(x-y)\\
&\{g_1(x), k(y)\} = k\delta(x-y) \qquad \{g_1(x), l(y)\} = l\delta(x-y)\\
&\{k(x), l(y)\} = (-\partial + 2g_1 + m_1 + m_2)\delta(x-y).
\end{aligned} \tag{4.8}$$

Using the transformation (4.1) and taking the above reduction into consideration, we conjecture that the Poisson bracket algebra (4.8) is related to the algebra $W_4^{(2)}$ (3.4) by the following transformation:

$$\begin{aligned}
&u = g_1 m_1 + ks - m_{1x} \qquad v = ln - m_2(m_1 + m_2 + g_1) - m_{2x}\\
&q = g_1 n + km_2 - n_x \qquad r = lm_1 - s(g_1 + m_1 + m_2) - s_x\\
&w = k + n \qquad p = l + s \qquad H = g_1 + m_1.
\end{aligned} \tag{4.9}$$

This conjecture can be verified by tedious but straightforward calculation.

As above, this realization does not qualify as a free field realization and we need to simplify (4.8) further to reach such a position. Our observation is that the following coordinate transformation $(m_1, m_2, n, s, g_1, k, l) \to (\hat{m}_1, \hat{m}_2, \hat{n}, \hat{s}, \hat{g}_1, \hat{k}, \hat{l})$

$$\begin{aligned}
&\hat{m}_1 = m_1 + m_2 \qquad \hat{m}_2 = m_1 - m_2 \qquad \hat{n} = n \qquad \hat{s} = s\\
&\hat{g}_1 = 2g_1 + m_1 + m_2 \qquad \hat{k} = k \qquad \hat{l} = l
\end{aligned} \tag{4.10}$$

brings (4.8) into the following form:

$$\{\hat{m}_1(x), \hat{m}_1(y)\} = -\partial\delta(x-y) \tag{4.11a}$$

$$\begin{aligned}
&\{\hat{m}_2(x), \hat{m}_2(y)\} = -2\partial\delta(x-y) \qquad \{\hat{m}_2(x), \hat{n}(y)\} = 2\hat{n}\delta(x-y)\\
&\{\hat{m}_2(x), \hat{s}(y)\} = -2\hat{s}\delta(x-y) \qquad \{\hat{n}(x), \hat{s}(y)\} = (-\partial + \hat{m}_2)\delta(x-y)
\end{aligned} \tag{4.11b}$$

$$\begin{aligned}
&\{\hat{g}_1(x), \hat{g}_1(y)\} = -2\partial\delta(x-y) \qquad \{\hat{g}_1(x), \hat{k}(y)\} = 2\hat{k}\delta(x-y)\\
&\{\hat{g}_1(x), \hat{l}(y)\} = -2\hat{l}\delta(x-y) \qquad \{\hat{k}(x), \hat{l}(y)\} = (-\partial + \hat{g}_1)\delta(x-y).
\end{aligned} \tag{4.11c}$$

We see that the above algebra consists of three closed subalgebras: a $U(1)$ current $\hat{m}_1$ and two copies of $sl(2)$ current algebra (4.11b, c). Once again, we may use the Wakimoto construction directly for the algebra (4.11a, b, c). It reads as

$$\begin{aligned}
&\hat{m}_1 = \theta \qquad \hat{m}_2 = \sqrt{2}\theta_1 + 2\theta_2\theta_3 \qquad \hat{n} = -\theta_2^2\theta_3 + \theta_{2x} - \sqrt{2}\theta_1\theta_2 \qquad \hat{s} = \theta_3\\
&\hat{g}_1 = \sqrt{2}\vartheta_1 + 2\vartheta_2\vartheta_3 \qquad \hat{k} = -\vartheta_2^2\vartheta_3 + \vartheta_{2x} - \sqrt{2}\vartheta_1\vartheta_2 \qquad \hat{l} = \vartheta_3
\end{aligned} \tag{4.12}$$

and the final algebra in coordinates $(\theta, \theta_1, \theta_2, \theta_3, \vartheta, \vartheta_1, \vartheta_2, \vartheta_3)$ is

$$\begin{aligned}
&\{\theta(x), \theta(y)\} = \{\vartheta(x), \vartheta(y)\} = -\partial\delta(x-y)\\
&\{\theta_1(x), \theta_1(y)\} = \{\vartheta_1(x), \vartheta_1(y)\} = -\partial\delta(x-y)\\
&\{\theta_2(x), \theta_3(y)\} = \{\vartheta_2(x), \vartheta_3(y)\} = -\delta(x-y).
\end{aligned} \tag{4.13}$$

Then, the composition of (4.9) and (4.10) with (4.12) supplies us the free field realization for the $W_4^{(2)}$ algebra (3.4).

Remarks.

(1) As we can see, the first step of the construction is systematic while the remaining steps are *ad hoc*. Thus, it is interesting to find an entire systematic method to rediscover the above results.

(2) This construction provides us with, as a by-product, a new proof of the Hamiltonian nature of the structure (3.4).

(3) The modified hierarchies for each set of coordinates are easily calculated.

(4) With the free field realizations we may construct quantized algebras for the Poisson algebra (3.4) and (3.3).

5. Toda-type theories connected with $W_4^{(2)}$

In this section we shall construct the Toda-type theory connected with the $W_4^{(2)}$ algebra. Precisely speaking, we shall construct a Toda theory which corresponds to two copies of the $W_4^{(2)}$ algebra: one copy is holomorphic, the other is anti-holomorphic. The construction is based on the following observations. Recall that the W-basis of the holomorphic copy of $W_4^{(2)}$ used in [3] is arranged in the following Drinfeld–Sokolov gauge:

$$Q = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ T_1 & G^{(+)} & U & Z \\ Y & T_2 & G^{(-)} & -U \end{pmatrix}.$$

Similarly we can have a W basis of the anti-holomorphic copy of $W_4^{(2)}$ which can also be arranged into the Drinfeld–Sokolov gauge

$$\bar{Q} = \begin{pmatrix} 0 & 0 & \bar{T}_1 & \bar{Y} \\ 0 & 0 & \bar{G}^{(+)} & \bar{T}_2 \\ -1 & 0 & \bar{U} & \bar{G}^{(-)} \\ 0 & -1 & \bar{Z} & -\bar{U} \end{pmatrix}.$$

Let g be the solution of the following linear systems:

$$\partial_+ g + Qg = 0 \qquad \partial_- g + g\bar{Q} = 0.$$

We can easily see that the matrix g can be realized by the matrix elements

$$g_a^b = \sum_i f_a^i \bar{f}_i^b$$

where f_i^j and $\bar{f}_i^j$ satisfy

$$\partial_x f_a^j = -f_{a+2}^j \qquad \partial_x \bar{f}_j^b = -\bar{f}_j^{b+2} \qquad a = 1, 2$$

and $\bar{f}_i^b$ have a similar property. Define the main diagonal subdeterminants Δ_a of the matrix g, i.e.

$$\Delta_a = \begin{vmatrix} g_1^1 & \cdots & g_1^a \\ \vdots & & \vdots \\ g_a^1 & \cdots & g_a^a \end{vmatrix}$$

and, in particular, $\Delta_0 \equiv 1$, we can prove, by tedious but direct calculations, that the matrix T with the elements (here $\Delta_a(i, j)$ denotes the algebraic co-minor of Δ_a with respect to g_i^j)

$$T_a^b \equiv \sqrt{\frac{\Delta_{a-1}}{\Delta_a}} \sum_{l=1}^{a} \frac{\Delta_a(l, a)}{\Delta_{a-1}} f_a^b$$

satisfy the equations

$$\partial_\pm T = \pm(\tfrac{1}{2}\partial_\pm\Phi + \exp(\mp\tfrac{1}{2}\,\mathrm{ad}\,\Phi)(\Psi_\pm + \mu_\pm))T \tag{5.1}$$

where we have used the following abbreviations of notation:

$$\begin{aligned}
&\Phi = \sum_{i=1}^{3}\phi^i H_i \qquad \phi^a = \ln\Delta_a \\
&\Psi_+ = \sum_{j=1}^{3}\sum_{i=1}^{4}\mathrm{sign}(i-j)\psi_i^+ A_{ij}E_j \qquad \psi_a^+ = \frac{\Delta_{a+1}(a, a+1)}{\Delta_a} \\
&\Psi_- = \sum_{j=1}^{3}\sum_{i=1}^{4}\mathrm{sign}(i-j)\psi_i^- A_{ij}F_j \qquad \psi_a^- = \frac{\Delta_{a+1}(a+1, a)}{\Delta_a}.
\end{aligned} \tag{5.2}$$

H_i, E_i and F_i are the standard Chevalley generators of the Lie algebra A_3 written in the defining representation, A is the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and $\mu_\pm$ are defined as

$$\mu_+ = \tfrac{1}{2}\sum_{i,j=1}^{3}[E_i, E_j] \qquad \mu_- = -\tfrac{1}{2}\sum_{i,j=1}^{3}[F_i, F_j].$$

Equation (5.1) can be viewed as the Lax pair of $W_4^{(2)}$ Toda theory, with the explicit solution of the Toda fields given by equation (5.2). The Toda field equation can be easily obtained from the compatibility condition of the Lax pair (5.1). The result reads

$$\begin{aligned}
&\partial_+\partial_-\Phi + [\mathrm{e}^{\mathrm{ad}\,\Phi}(\Psi_-), \Psi_+] + [\mathrm{e}^{\mathrm{ad}\,\Phi}(\mu_-), \mu_+] = 0 \\
&\partial_-\Psi_+ - [\mu_+, \mathrm{e}^{\mathrm{ad}\,\Phi}(\Psi_-)] = 0 \\
&\partial_+\Psi_- - [\mathrm{e}^{\mathrm{ad}\,\Phi}(\Psi_+), \mu_-] = 0.
\end{aligned}$$

In terms of the component fields, the above equations read (K is the Cartan matrix of A_3)

$$\begin{aligned}
&\partial_+\partial_-\phi^j - \sum_{i,k=1}^{4}\mathrm{sign}(i-j)\,\mathrm{sign}(k-j)\psi_i^+ A_{ij}\psi_k^- A_{kj}\omega^j + \sum_{l=1 l\neq j}^{3}\omega^l\omega^j K_{ij} = 0 \\
&\partial_-\psi_j^+ - \sum_{k=1}^{4}\mathrm{sign}(k-j)\psi_k^- A_{kj}\omega^j = 0 \\
&\partial_+\psi_j^- - \sum_{k=1}^{4}\mathrm{sign}(k-j)\psi_k^+ A_{kj}\omega^j = 0 \\
&\omega^j \equiv \exp(-\sum_{i=1}^{3}\phi^i K_{ij}) \qquad (j = 1, 2, 3) \\
&\partial_-\psi_4^+ = \partial_+\psi_4^- = 0.
\end{aligned}$$

Remarks.

(1) The above construction of Toda-type theory is essentially an extension of the technique of W surfaces, which was first developed by Gervais and Matsuo [13] in the

standard W_N cases. Thus the construction given here not only presents the $W_4^{(2)}$ Toda equation but also the $W_4^{(2)}$ surface in the sense of [13].

(2) Toda-type equations associated with general $W_N^{(2)}$ algebras have already been studied by one of the authors (LC) and collaborators in several papers [14]. However, those equations restricted to the case of $N = 4$ lack the fields $\psi_4^{\pm}$, thus do not really correspond to $W_4^{(2)}$ algebra. The present equations overcome this shortcoming.

(3) The functions f_a^i and $\bar{f}_i^a$ can be shown to satisfy two commuting families of classical exchange algebra for $a = 1, 2$. For example, the holomorphic family of exchange algebra reads

$$\begin{aligned}\{f_a^i(x), f_b^j(y)\} = -\tfrac{1}{8} f_a^i(x) f_b^j(y)\,\mathrm{sign}(x-y) + f_a^j(x) f_b^i(y)[\theta(i-j)\theta(x-y)\\ -\theta(j-i)\theta(y-x)] \qquad a, b = 1, 2\end{aligned} \tag{5.3}$$

where

$$\theta(a-b) \equiv \begin{cases} \frac{1}{2} & (a-b=0) \\ 0 & (a-b<0) \\ 1 & (a-b>0) \end{cases} \qquad \mathrm{sign}(a-b) = \theta(a-b) - \theta(b-a). \tag{5.4}$$

Such exchange algebras can be used to reconstruct $W_4^{(2)}$ algebra since one can always write the W basis of $W_4^{(2)}$ algebra in terms of appropriate determinants consisting of the above functions. This construction of W algebras can also be extended to any classical $W_N^{(l)}$ algebra [8]. Since the classical exchange algebra is the origin of the quantum group, it may also be possible to relate quantum W algebras and quantum groups in terms of a quantized version of such constructions.

(4) The canonical Poisson structure for the $W_4^{(2)}$ Toda fields can also be obtained from the exchange relation (5.3) and the explicit solution (5.2) of the field equations.

6. Conclusions

In this paper we have constructed both the integrable evolution equations and the corresponding Toda theory associated with the $W_4^{(2)}$ algebra. Miura maps are presented in connection with the $W_4^{(2)}$ evolution equations, which in turn give a free field realization of $W_4^{(2)}$ algebra.

We have shown that the $W_4^{(2)}$ algebra, for which the exchange of evolution parameter approach failed, can be studied through the matrix Lax operator. This may be true for the general classical $W_n^{(l)}$ and deserves further consideration. Also, though the problem considered here is only a specific case of the W-algebra–evolution equation–Toda system connections, the constructions presented here again assure the widely adopted conjecture that, given a W algebra, there must exist an associated system of evolution equations and a corresponding Toda theory.

Besides what has been considered in the main text of this paper, we would like to mention that there are still some unsolved problems, such as the connection between the variables appearing in the evolution equations and the Toda fields. As the $W_4^{(2)}$ algebra is much more complicated than the standard W_N series, one should reasonably feel that such connections are not so straightforward as in the standard case.

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